A scaling-limit approach to the theory of laser transition

Paul Gartner

Institute for Theoretical Physics,

University of Bremen,

28334 Bremen, Germany

and

National Institute of Materials Physics,

Bucharest-Măgurele, Romania[*]

(Dated: February 25, 2013)

Abstract

The conditions for the appearance of a sharp laser transition are formulated in terms of a scaling limit, involving vanishing cavity loss and light-matter coupling, $\kappa \to 0$, $g \to 0$, such that g^2/κ stays finite. It is shown analytically that in this asymptotic parameter domain, and for pump rates above the threshold value, the photon output becomes large in a sense that is specified, and the photon statistics becomes strictly Poissonian. Numerical examples for the case of a two-level and a three-level emitter are presented and discussed in relation to the analytic result.

[*] gartner@itp.uni-bremen.de

I. INTRODUCTION

An early preoccupation of laser theory was the analogy between the onset of lasing and phase transitions[1–3]. Within the cavity QED models the problem became increasingly accessible to accurate numerical treatments, and evidence that an abrupt change of regime takes place has accumulated. In parallel, approximate analytic considerations also argued in the same sense.

A particularly illuminating case is the so-called random injection model of Scully and Lamb[4], which has the advantage of addressing directly the cavity mode statistics. The model was extensively studied [5] and is by now textbook material[6, 7]. In emitter-plus-mode models the situation is more complicated because the photonic state information has first to be extracted, by eliminating the emitter degrees of freedom. This procedure can be carried out[8] in the case of single emitters with few (usually only two) states. Then information about the photon statistics can be obtained, either from the numerics or, using simplifying approximations, analytically too[9–14].

In this context the seminal paper by Rice and Carmichael, Ref. 11, has drawn the attention on the necessity of a limiting procedure for obtaining a sharp transition, with a precisely defined threshold. The analogy with the thermodynamic limit in the theory of phase transition was invoked. It was argued that for the lasing transition the limit involves the β-factor going to 0, together with the cavity loss κ so that the ratio β/κ remains finite. In this limit, and for a pump rate exceeding a threshold value, the number of photons N becomes infinite, generating "an explosion of stimulated emision"[11], and the appropriate object of study is the rescaled value βN, which stays finite.

In the present paper we show that a similar scaling limit is needed for an abrupt onset of lasing, but we use in our formulation the more ubiquitous Jaynes-Cummings (JC) coupling constant g, instead of the β-factor. The latter is proportional to g^2 and indeed our scaling procedure implies $\kappa \to 0$, $g \to 0$ so that g^2/κ is finite. In this scaling limit and above the threshold one obtains $N \to \infty$ but such that κN remains finite. Moreover we are able to prove not only that the transition becomes abrupt, but also that the photon statistics above the threshold turns exactly Poissonian. The proof is analytic and does not rely on approximations. Also, analytic expression for relevant data, like the threshold pump rate, level occupancies and photon output in the lasing regime, are obtained. Numerical results

are shown as illustration of the statements.

II. THE MODEL AND STATEMENT OF THE RESULT

Single emitter lasers are commonly described as embedded JC systems. In other words, two emitter states, either quantum dot or atomic configurations, interact with the cavity mode via the JC Hamiltonian

$$H_{JC} = g\, b^{\dagger} \left|2\right\rangle \left\langle 1\right| + g\, b \left|1\right\rangle \left\langle 2\right| \ , \tag{1}$$

in the presence of, possibly, other states. Here $b, b^{\dagger}$ are the photonic operators and the emitter states are denoted by $|i\rangle$. In particular $|1\rangle$ and $|2\rangle$ specify the upper and the lower laser state, respectively. We assume that the cavity mode is resonant with the laser transition. Dissipation effects are included in the master equation for the density operator ρ (in the interaction picture and with $\hbar = 1$)

$$\frac{\partial}{\partial t}\rho = -i\left[H_{JC}, \rho\right] + \mathcal{L}(\rho) \tag{2}$$

by the Lindblad terms

$$\mathcal{L}(\rho) = \frac{\kappa}{2}\left[2b\rho b^{\dagger} - b^{\dagger} b\rho - \rho b^{\dagger} b\right] + \sum_{(i,j)} \frac{\gamma_{ij}}{2}\left[2\sigma_{ij}\rho\sigma_{ij}^{\dagger} - \sigma_{ij}^{\dagger}\sigma_{ij}\rho - \rho\sigma_{ij}^{\dagger}\sigma_{ij}\right] \ , \quad \sigma_{ij} = \left|i\right\rangle\left\langle j\right| \ . \tag{3}$$

A central role here is played by the first term, describing the cavity losses at a rate κ, while the second term summarizes transition processes from states $|j\rangle$ to $|i\rangle$ at rates γ_{ij}. Lowering σ_{ij} operators and their hermitian conjugates $\sigma_{ij}^{\dagger}$ give rise to Lindblad terms accounting for relaxation, but raising terms are considered as well, to simulate incoherent pumping [15], and the corresponding rate will then be denoted by P.

The master equation is solved in time until a steady-state solution is reached, from which data concerning level occupancies and photon statistics can be extracted, as function of the pumping rate and other parameters. With this information at hand one can address the problem of laser transition: how to identify it and what are the conditions for its appearance.

The answer to the first question is that the lasing regime is defined by accumulation of a large number of photons in the cavity, the statistics of their number n obeying a Poissonian law

$$\rho_{n,n} = \sum_{i} \rho_{n,n}^{i,i} = \frac{\lambda^{n}}{n!} e^{-\lambda} \quad , \quad \text{for all } \ n \ . \tag{4}$$

Equivalently, the Poisson statistics amounts to the requirement that the normal-ordered expectation values $p_n = \langle b^{\dagger\, n} b^n \rangle$ depend exponentially on n, $p_n = \lambda^n$, or that the zero time-delay n-th order correlation functions $g^{(n)} = p_n/p_1^n$ all become equal to 1. The problem is that, on the one hand, the large number condition is imprecise (how large is large?) and, on the other hand, no matter which of these criteria for Poisson statistics one chooses to apply, one has to check an infinite set of equalities. This is practically impossible, either experimentally or numerically. This is why it is often encountered in the literature that one limits oneself to simpler lasing criteria, like $N > 1$ and $g^{(2)} = 1$.

The aim of the present paper is to formulate and prove the conditions under which strict Poissonian statistics is generated and at the same time to specify in what sense the photon output becomes large. Essentially these conditions involve the limit $\kappa \to 0$ and simultaneously $g \to 0$, but with the JC coupling parameter scaling like $\sqrt{\kappa}$ so that the ratio g^2/κ remains finite. The necessity of a certain limiting procedure for obtaining a well-defined transition to a purely Poissonian statistics was recognized long ago[11] and is analogous to the thermodynamic limit in the theory of phase transitions. The precise formulation of our statement runs as follows:

a. Rescale κ as $\varepsilon\,\kappa$ and g as $\sqrt{\varepsilon} g$. Then, in the limit $\varepsilon \to 0$ and for the pump rate above a threshold value, the average photon number $N = p_1$ tends to infinity in such a way that the product εN remains finite.

b. In the same limit and above the threshold, the rescaled expectation values $\tilde{p}_n = \varepsilon^n\, p_n$ remain finite for all n, with their limit values obeying an exponential law $\tilde{p}_n = \tilde{p}_1^n$. Equivalently, all the correlation functions become $g^{(n)} = 1$. Below the threshold the increase in photon production is not sufficient, leading to $\tilde{p}_n = \varepsilon^n\, p_n \to 0$ for all $n \geq 1$, and the correlation functions are in general different from 1.

c. In this scaling limit the transition between the two regimes becomes sharp, with a well-defined threshold point.

Several comments are in order. The model parameters, g and κ,that are rescaled correspond to the photon source and sink, respectively. They are present in all laser models and therefore the formulation of the scaling limit in these terms makes sense in various situations. The limit of small cavity losses (good cavity) pleads in favour of photon accumulation, but with the simultaneous diminishing of the production rate, proportional to g^2, their overall increase is not a foregone conclusion. The statement (a) above spells out what is meant by

a large photon number, namely that it should scale like $1/\kappa$ to compensate for the reduction of the rate of escape from the cavity. Thus, even for a cavity quality factor Q close to infinity there is still light coming out from the device.

In situations as those described by incoherent excitation, at high pump rates the phenomenon of self-quenching[16] might occur. Beside producing population inversion, the pump can destroy coherence between the laser levels and inhibit the transition. This plays against lasing and therefore one may encounter a double transition, one at the onset of lasing and the other when lasing becomes quenched. In such cases the lasing regime takes place in a given interval of the excitation rates, limited below by the threshold value and above by the self-quenching $P_{thr} < P < P_{sq}$. The endpoints of this interval depend on the model parameters. Accordingly, it may occur that the interval shrinks to zero and then no transition takes place. Such situations will also be discussed below.

In what follows we will first bring numerical evidence in favor of the scaling limit result. We illustrate the situation with calculations performed on a two-level and on a three-level model. In both cases the scaling limit tendencies are quite clear. Still, numerical statements do not amount to a proof, which can only rely on an analytic argument. We are able to formulate an analytic derivation of the scaling limit result in the two-level case (see Sec. III). Also, an analytic proof is available[17] for the random injection model[4, 5], which does not belong to the class defined by Eqs. (2),(3). We believe that all these results speak in favor of a wider generality of the scaling limit statement.

A. Numerical results

Steady-state results are obtained from the long-time limit of the master evolution. Particularly indicative of a transition is the behavior of the population inversion $w = \langle\, |1\rangle\langle 1| - |2\rangle\langle 2|\,\rangle$ as a function of the pump. Depending on the parameters, one clearly detects two types of behavior[12, 14] illustrated in Fig.1(a) [18]. The plots correspond to a two-level emitter in which the pumping is described by the raising $|2\rangle \to |1\rangle$ Lindblad operator with the rate $P = \gamma_{12}$, while for the rate of loss to non-lasing modes we use the notation $\gamma_{21} = \gamma$. For one set of parameters the curve is strictly concave, while for the other there appears an almost perfectly linear shortcut[9, 17, 19] separating two concave regions. This clearly suggests that in the latter case an abrupt change of regime is taking place, in

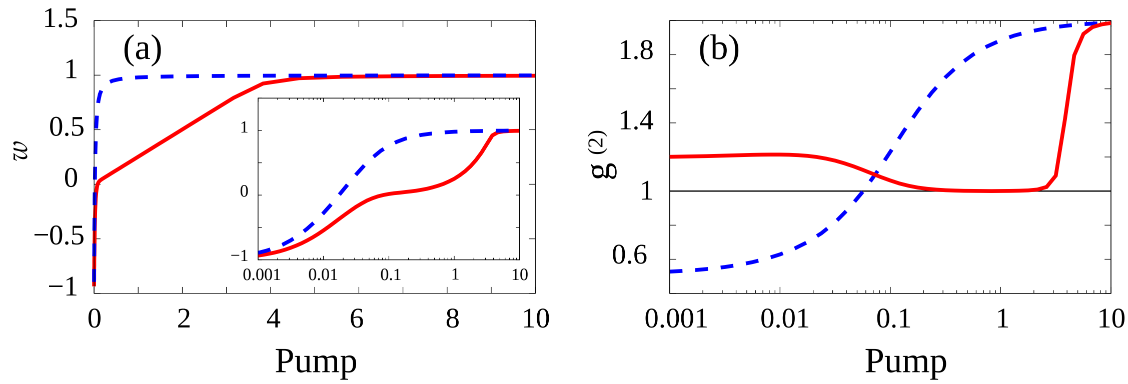

FIG. 1. (a) Population inversion w in linear and (inset) semilogarithmic plot, and (b) second order correlation function $g^{(2)}$ for a two-level model. The parameters are, red (solid) line: $\gamma = 0.02$, $g = 0.1$ and $\kappa = 0.01$, blue (dotted) line: $\gamma = 0.01$, $g = 0.01$ and $\kappa = 0.02$.

a given P-interval. It is also obvious that the appearance of the transition is conditioned by certain parameter values. By examining Fig.1(b), we see that this linear segment (which in a semilogarithmic representation appears as an additional convex region, see inset of Fig.1a), corresponds to $g^{(2)}$ being very close to unity, which is characteristic for coherent light. Therefore it is natural to assume that we are in the presence of the lasing regime. Moreover, the fact that for large P-values the linear behavior disappears is consistent with the inhibition of lasing by self-quenching.

Zooming in on the leftmost point of the linear interval, as in Fig.2, it is seen that the transition becomes more and more abrupt as the scaling parameter becomes smaller, in accordance with the scaling limit statement. This is seen both in the panel (a) of Fig.2, which refers to the two-level model discussed in Fig.1, and in the panel (b), which shows the result for a three-level emitter. In this latter case the pump is raising the system from the lower laser state $|2\rangle$ to a third state $|3\rangle$ ($P = \gamma_{32}$), wherefrom it relaxes to the upper laser state $|1\rangle$ with the rate $\gamma' = \gamma_{13}$. As before $\gamma_{21} = \gamma$.

The same tendency is seen in plots of $g^{(2)}$, which becomes closer and closer to the coherent light value $g^{(2)} = 1$, as the scaling parameter becomes smaller, both for the two- and for the three-level case, as illustrated in Fig.3.

Finally, in Fig.4 we show numerical results for the photon output. It is seen that in the lasing interval the rescaled photon numbers $\tilde{N} = \varepsilon N$ have practically reached their limit

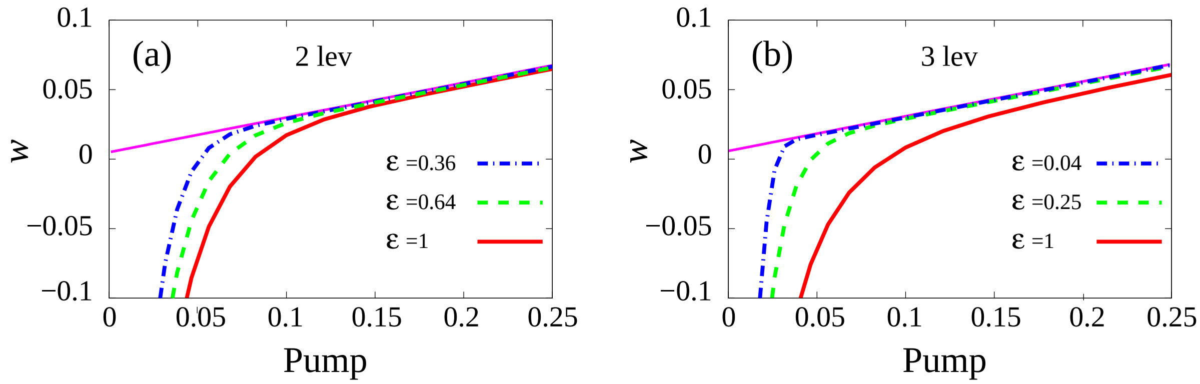

FIG. 2. Scaling parameter dependence of the population inversion w for (a) the two-level emitter with parameters: $\gamma = 0.02$, $g = \sqrt{\varepsilon}\,0.1$ and $\kappa = \varepsilon\,0.01$ and (b) the three-level emitter with parameters: $\gamma = 0.02$, $\gamma' = 0.05$, $g = \sqrt{\varepsilon}\,0.1$ and $\kappa = \varepsilon\,0.01$. Thin solid lines correspond to the analytic result Eq. (17).

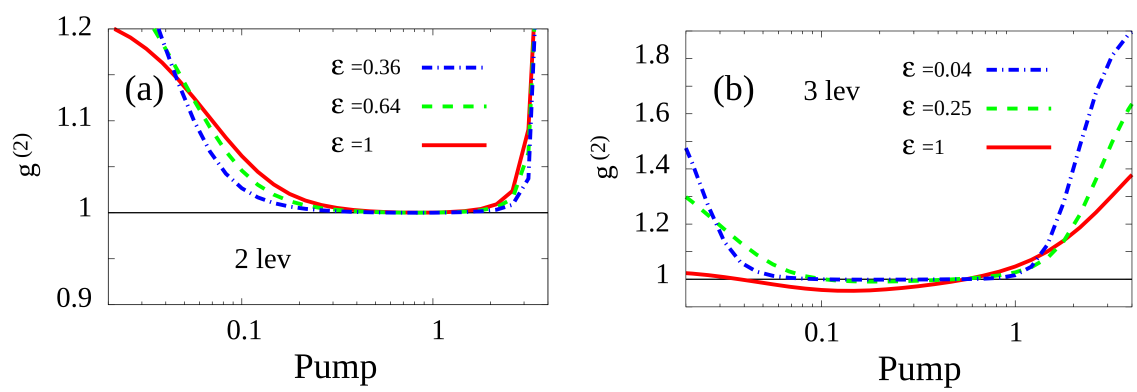

FIG. 3. Scaling parameter dependence of $g^{(2)}$ for the same parameters as in Fig.2.

values, given by Eqs. (18) and (19), also plotted in the figure. This means that indeed, N grows like $1/\varepsilon$, in accordance with the scaling statement.

The agreement seen in this section between the numerical data and the scaling limit results can hardly be accidental. Values of $g^{(2)}$ close to unity suggest that in the interval of intermediate pump strengths the system operates in the lasing regime, and the large photon output speaks in favor of this supposition too. Nevertheless, it is not at all clear yet how the linear dependence of the population inversion on the excitation is in any way linked to lasing. The analytic arguments of the next section will prove that, indeed, the two are

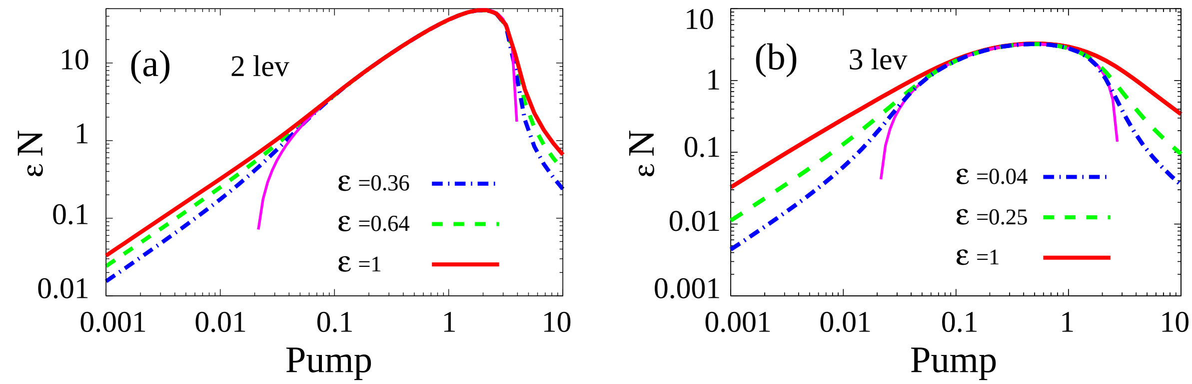

FIG. 4. Rescaled photon output for the two- and three-level emitter cases. The parameters are the same as in Fig.2. Thin solid lines correspond to Eq. (18) in panel (a), and to Eq. (19) in panel (b).

related and appear simultaneously.

III. THE TWO LEVEL LASER

A special feature of Eq. (2) is the fact that it provides a system of closed equations for a subclass of relevant density matrix elements. In this category, the elements which are diagonal in the emitter states $|i\rangle$ are also diagonal in the photon number n, $\rho^{i,i}_{n,n}$, and the only off-diagonal elements are of the form $\rho^{1,2}_{n,n+1}$ and their complex conjugates $\rho^{2,1}_{n+1,n}$. This is a consequence of fact that the JC Hamiltonian conserves the excitation number $|1\rangle\langle 1| + b^\dagger b$.

We consider here the case of an emitter consisting of only the two laser states $|1\rangle$ and $|2\rangle$. The Lindblad terms describe, beside the cavity losses, the spontaneous emission into non-lasing modes and the pumping, with the rates $\gamma = \gamma_{21}$ and $P = \gamma_{12}$, respectively. The master equation implies an infinite set of equations of motion for expectation values. The above-mentioned limitation for the density matrix elements involved, translates into a closed system of equations of motion for a reduced number of relevant expectation values. These are

$$\begin{aligned}
c_n &= \left\langle |1\rangle\langle 1|\; b^{\dagger\, n} b^n \right\rangle \,, && n = 0,1,2\ldots\,, \\
v_n &= \left\langle |2\rangle\langle 2|\; b^{\dagger\, n} b^n \right\rangle \,, && n = 0,1,2\ldots \quad \text{and} \\
t_n &= -ig\left\langle |2\rangle\langle 1|\; b^{\dagger\, n} b^{n-1} \right\rangle \,, && n = 1,2,3\ldots
\end{aligned} \tag{5}$$

Obviously, the average population of the upper (lower) level is given by c_0 (v_0), which obey $c_0 + v_0 = 1$. Of a special interest for the photon statistics are the expectation values $p_n = \left\langle b^{\dagger n} b^n \right\rangle = c_n + v_n$, and in particular the average photon number p_1, which is also denoted by N. The imaginary prefactor in the definition of the multi-photon assisted polarization t_n makes it a real-valued quantity.

The equation of motion for the expectation value of a given operator A can be obtained from the master equation via

$$\frac{\partial}{\partial t} \langle A \rangle = \mathrm{Tr} \left\{ A \frac{\partial}{\partial t} \rho \right\} = -i \langle [A, H_{JC}] \rangle + \frac{\kappa}{2} \left\langle \left[b^\dagger, A \right] b + b^\dagger \left[A, b \right] \right\rangle + \sum_{(i,j)} \frac{\gamma_{ij}}{2} \left\langle \left[\sigma_{ij}^\dagger, A \right] \sigma_{ij} + \sigma_{ij}^\dagger \left[A, \sigma_{ij} \right] \right\rangle . \quad (6)$$

For those in Eq. (5) this leads to the following equations of motion and the corresponding steady-state conditions [8, 12, 17]

$$\frac{\partial}{\partial t} c_n = -(n\kappa + \gamma) c_n + P v_n - 2 t_{n+1} = 0 , \quad (7)$$

$$\frac{\partial}{\partial t} v_n = \gamma c_n - (n\kappa + P) v_n + 2 t_{n+1} + 2n t_n = 0 , \quad (8)$$

$$\frac{\partial}{\partial t} t_n = g^2 c_n + g^2 n c_{n-1} - g^2 v_n - \frac{(2n-1)\kappa + P + \gamma}{2} t_n = 0 . \quad (9)$$

By adding Eqs. (7) and (8) one obtains the steady-state balance relation between the losses from the cavity and its feeding through the photon-assisted polarization

$$\kappa\, p_n = 2\, t_n , \quad n \geq 1 . \quad (10)$$

Using this condition and Eq. (7) with $n = 0$, one obtains $P\, v_0 = \gamma\, c_0 + \kappa\, N$, which allows to express the steady-state level occupancies in terms of the photon output N

$$c_0 = \frac{P - \kappa\, N}{P + \gamma} , \quad v_0 = \frac{\gamma + \kappa\, N}{P + \gamma} . \quad (11)$$

The unknowns c_n and v_n can be eliminated from Eqs. (7) and (8) in favor of t_n and, using again the balance condition Eq. (10), one is lead[8] to a three-term recursion equation for the photonic quantities p_n

$$A_n\, p_{n+1} + B_n\, p_n - C_n\, p_{n-1} = 0 , \quad n \geq 1 , \quad (12)$$

with

$$
\begin{aligned}
A_n &= \frac{2\,\kappa}{n\,\kappa + P + \gamma}\,, \\
B_n &= \frac{n\,\kappa - P + \gamma}{n\,\kappa + P + \gamma} + \frac{n\,\kappa}{(n-1)\,\kappa + P + \gamma} + \kappa\,\frac{(2n-1)\,\kappa + P + \gamma}{4g^2}\,, \\
C_n &= \frac{n\,P}{(n-1)\,\kappa + P + \gamma}\,.
\end{aligned}
\tag{13}
$$

By using the well-established connection between three-term recursion problems and continued fractions[20], Eq. (12) allows for obtaining directly steady-state values, without resorting to the time evolution[17]. The convergence of the continued-fraction solution is very good in all points, except the intermediate pumping region where the transition takes place and the population inversion becomes linear.

In that interval an excellent agreement with the numerical solution can be obtained by the following simple ansatz: Assume that (i) Eq. (12) is valid for $n = 0$ too, and (ii) the last term $C_0\,p_{-1}$ takes in this case the value 0, so that one has

$$A_0\,p_1 + B_0\,p_0 = 0\,. \tag{14}$$

With $p_0 = 1$ and the values A_0 and B_0 as in Eq. (13) one obtains for the average photon number

$$N = -\frac{B_0}{A_0} = \frac{P-\gamma}{2\kappa} - \frac{(P+\gamma)(P+\gamma-\kappa)}{8g^2}\,. \tag{15}$$

Using this in Eq. (11) leads for the population inversion $w = c_0 - v_0$ to

$$w = \kappa\,\frac{P+\gamma-\kappa}{4g^2}\,, \tag{16}$$

showing that the linear P-dependence of w is a direct consequence of the ansatz.

In the scaling limit, $\kappa \to \varepsilon\kappa$, $g \to \sqrt{\varepsilon}g$ with $\varepsilon \to 0$, these results become

$$w = \kappa\,\frac{P+\gamma}{4g^2}\,, \tag{17}$$

for the population inversion and

$$\tilde{N} = \frac{P-\gamma}{2\kappa} - \frac{(P+\gamma)^2}{8g^2}\,. \tag{18}$$

for the rescaled photon output $\tilde{N} = \varepsilon N$.

Similar results can be obtained for the three-level model (details of the calculations are left for a future publication) with the conclusion that in the scaling limit the population inversion behaves in as in Eq. (17) above, while the rescaled photon population obeys

$$\tilde{N} = \frac{(P-\gamma)\gamma'}{\kappa(P+2\gamma')} - \frac{(P+\gamma)(P\gamma+P\gamma'+\gamma\gamma')}{4g^2(P+2\gamma')} \, . \tag{19}$$

Note that in the limit of large γ', that is for very fast $|3\rangle \to |1\rangle$ relaxation, one recovers the two-level result, Eq. (18), as expected. The agreement of these expressions, Eqs. (17–19), with the numerical simulations is illustrated in Figs. 2 and 4.

It is clear that the ansatz makes sense only in the interval of P values for which $N \geq 0$. The first term in Eq. (15) is positive if P is not too small. On the other hand, for large P values, the second, negative term becomes dominant, so that the positivity condition can hold only for a finite interval.

Needless to say, the very good agreement between the ansatz and the numerical results (in the interval where the former makes sense) does not constitute a valid proof of the former. It is not immediately obvious why Eq. (12) should hold for $n = 0$. Neither can one use $C_0 = 0$ as an argument to replace the last term $C_0\, p_{-1}$ with 0[19], because it also contains the ill-defined p_{-1}. In order to prove the result one has to show first that, indeed, the three-term recursion relation can be extended for $n = 0$, identifying in the process the quantity appearing in the role of $C_0\, p_{-1}$. In a second step, one has to show that, in certain conditions, this quantity does vanish, as required by the ansatz.

The first step of this program is the easier part. It relies on the Glauber-Sudarshan (GS) $\mathcal{P}$-representation for the photonic density operator [21] as an integral over the complex plane of coherent states

$$\rho = \int |\alpha\rangle\, \mathcal{P}(\alpha)\, \langle\alpha|\, \frac{\mathrm{d}^2\alpha}{\pi} \, . \tag{20}$$

Since there is no preferred phase angle in the theory (the reduced photonic density operator, obtained by tracing out the emitter indices, is diagonal in the photon number basis) the $\mathcal{P}$-function depends only on $s = |\alpha|^2$, $\mathcal{P}(\alpha) = \mathcal{P}(s)$. Then, the normal-ordered photonic expectation values p_n turn out to be moments of the quasi-distribution defined by $\mathcal{P}$

$$p_n = \int_0^\infty s^n \mathcal{P}(s)\, \mathrm{d}s \, . \tag{21}$$

Note, for further reference, that the Poissonian statistics, characterized by $p_n = \lambda^n$ would correspond to a sharp peak in the GS function $\mathcal{P}(s) = \delta(s-\lambda)$ with $\lambda > 0$.

The $\mathcal{P}$-representation contains, in principle, the same information as the density operator, but here we take advantage that it allows a natural extension for the definition of expectation values. A case in point is p_n which, using Eq. (21), becomes well-defined even for n taking continuous, (not just integer) positive values. Using the differential equations which translate the master equation Eq. (2) into the $\mathcal{P}$-representation formalism [21], one recovers the three-term recursion formula with a continuous index, including the result for index zero[17]. The latter can also be obtained by taking the limit $n \to 0$ in Eq (12) and using for evaluating $C_n\, p_{n-1}$

$$np_{n-1} = \int_0^\infty (s^n)'\mathcal{P}(s)\,\mathrm{d}s = -\int_0^\infty s^n\mathcal{P}'(s)\,\mathrm{d}s \xrightarrow{n\to 0} \mathcal{P}(0)\ . \tag{22}$$

It is now obvious that the third term in the recursion relation Eq. (12) for $n = 0$ is proportional to $\mathcal{P}(0)$ and therefore the validity of the ansatz is equivalent to the requirement that $\mathcal{P}(s)$ vanishes for $s = 0$.

The second task is to establish the conditions when the vanishing takes place. This is the central point not only in the justification of the ansatz, but also in the proof of the scaling limit result. The latter amounts to showing that in this limit the $\mathcal{P}$-function becomes a δ-distribution concentrated on a positive value, and this obviously entails that indeed, its value at the origin becomes vanishingly small.

To this end we look at the differential equation obeyed by $\mathcal{P}(s)$, and which translates the master equation into the language of the $\mathcal{P}$-representation. We summarize here the main steps, the details can be found in [17]. To start with, the density matrix of our problem has a two-by-two block structure, corresponding to the two levels of the emitter. Correspondingly one has four $\mathcal{P}$-functions placed in a matrix $\mathcal{P}_{i,j}(s),\ i,j = 1,2$ and the photonic function we are interested in is obtained by tracing out the emitter-state indices $\mathcal{P} = \mathcal{P}_{1,1} + \mathcal{P}_{2,2}$. Using the rules for mapping the master equation for ρ into a Fokker-Planck equation for $\mathcal{P}$, [21] one is lead to a system of equations which is the counterpart of Eqs. (7–9). After eliminating $\mathcal{P}_{1,1}$ and $\mathcal{P}_{2,2}$ in favor of $\mathcal{P}$ we obtain a second-order differential equation for the latter.

This equation has then to be analyzed in the scaling limit. To simplify the notation we take κ itself as the scaling parameter which goes to 0, and impose the condition $g \to 0$ with g^2/κ fixed, by writing $g^2 = \tilde{g}^2\kappa$ and keeping $\tilde{g}^2$ constant. The rescaled expectation values $\tilde{p}_n = \kappa p_n$, which are the object of the scaling statement, can be obtained as moments of a

rescaled $\mathcal{P}$-function

$$\tilde{p}_n = \int_0^\infty \kappa^n s^n \mathcal{P}(s)\,\mathrm{d}s = \int_0^\infty t^n \tilde{\mathcal{P}}(t)\,\mathrm{d}t\ , \tag{23}$$

with $t = \kappa s$ and

$$\tilde{\mathcal{P}}(t) = \frac{1}{\kappa}\,\mathcal{P}\left(\frac{t}{\kappa}\right)\ . \tag{24}$$

If, indeed, the number of photons increases to infinity in the scaling limit, then the $\mathcal{P}$ quasi-distribution function moves its weight to larger and larger values and its moments cease to exist. In this situation only the rescaled function remains meaningful. Intuitively, according to Eq. (24), the graph of the rescaled function $\tilde{\mathcal{P}}(t)$ is obtained from that of the original $\mathcal{P}(s)$ by compressing the latter by a factor of $1/\kappa$ along the abscissa and expanding it by the same factor along the ordinate. This would bring the rescaled function to $\delta(t)$, in the limit $\kappa \to 0$, were it not for the opposite tendency of $\mathcal{P}(s)$ to move away from the origin, as discussed above. The net result of these competing trends is what one has to establish. It is easy to rewrite the differential equation obeyed by $\mathcal{P}(s)$ into the corresponding one for $\tilde{\mathcal{P}}(t)$, and retain in the coefficients only the dominant terms in the scaling parameter κ. The result is [17]:

$$\frac{t^2}{4\tilde{g}^2}\,\kappa^2\,\tilde{\mathcal{P}}'' - \left[\left(3\,\frac{\gamma+P}{8\tilde{g}^2}+1\right)t - \frac{1}{2}\,P\right]\kappa\,\tilde{\mathcal{P}}' + (t-\nu)\;\tilde{\mathcal{P}} = 0\ , \tag{25}$$

with ν an essential parameter in the discussion

$$\nu = \frac{P-\gamma}{2} - \frac{(P+\gamma)^2}{8\tilde{g}^2}\ , \tag{26}$$

whose P dependence is important and therefore sometimes emphasized by the notation $\nu(P)$. Note that ν is the same as the rescaled photon population κN, see Eq. (18), so that they change sign simultaneously.

The appearance of the small parameter κ along with the derivatives suggests a WKB approach to the $\kappa \to 0$ asymptotics of the solution. In other words one searches the solution, up to a normalization factor, in the form

$$\tilde{\mathcal{P}}(t) = \exp\left(-\frac{1}{\kappa}\,\varphi(t)\right)\ , \tag{27}$$

in which $\varphi(t)$ is taken in the leading, zeroth order in κ. It is clear that when κ gets smaller, the value of $\tilde{\mathcal{P}}(t)$ around the minimum of $\varphi(t)$ is greatly enhanced, in comparison with the values at other points which, in the view of normalization, become negligible. In the limit one obtains a δ-function concentrated at the minimum of $\varphi(t)$.

The equation obeyed by $\varphi(t)$ in the leading order has the form of a quadratic equation for its derivative

$$\frac{t^2}{4\tilde{g}^2}\,(\varphi')^2+\left[\left(3\,\frac{\gamma+P}{8\tilde{g}^2}+1\right)t-\frac{1}{2}\,P\right]\,\varphi'+[t-\nu(P)]=0\;. \tag{28}$$

Around $t=0$ one of the roots behaves like $\varphi'\sim 2\tilde{g}^2P/t^2$, i.e. $\varphi\sim -2\tilde{g}^2P/t$ which in Eq. (27) leads to a strongly singular solution. The regular one comes from the other root for which $\varphi'(0)=-2\nu(P)/P$.

Two cases arise, depending on the sign of $\nu(P)$: (i) As long as $\nu(P)$ is negative, $\varphi'(0)>0$, then $t=0$ is a minimum for $\varphi(t)$ and, according to the above discussion, $\tilde{\mathcal{P}}(t)$ tends to $\delta(t)$ in the scaling limit. (ii) When $\nu(P)$ becomes positive, φ' starts at $t=0$ with negative values and crosses the abscissa at $t=\nu$. Then $\varphi(t)$ has a local maximum at the origin and therefore the values of $\tilde{\mathcal{P}}(0)$ become vanishingly small in the limit $\kappa\to 0$. This is precisely the requirement for the ansatz to hold. The function $\tilde{\mathcal{P}}(t)$ is now concentrated at the point of minimum $t=\nu(P)$.

As a consequence, in the interval in which $\nu(P)$ is positive one has $\tilde{\mathcal{P}}(t)\to\delta(t-\nu)$ and all the rescaled expectation values become $\tilde{p}_n=\nu^n, n\geq 0$. Outside this interval $\tilde{\mathcal{P}}(t)\to\delta(t)$ and all $\tilde{p}_n=0$, except $\tilde{p}_0=p_0$, which is equal to 1 by definition. The change is abrupt and takes place at the interval endpoints defined by the quadratic equation $\nu(P)=0$. The condition for this equation to have real roots is $\tilde{g}^2\geq 2\gamma$, or $g^2\geq 2\kappa\gamma$, and then the roots are both positive

$$P_\pm=2\tilde{g}^2-\gamma\pm 2\tilde{g}\sqrt{\tilde{g}^2-2\gamma}\;. \tag{29}$$

The lowest one, $P_-=P_{thr}$, corresponds to the onset of lasing and the highest, $P_+=P_{sq}$, to self-quenching. The condition $\tilde{g}^2\geq 2\gamma$ distinguishes the two behaviors illustrated in Fig.1 because, when not fulfilled, no transition takes place. With this, the proof of the scaling limit is complete.

It is instructive to see the action of the scaling limit directly on the recursion relation Eq. (12). The essential point is the observation that the n-dependence of the coefficients A_n, B_n, C_n gradually disappears in the limit $\kappa\to 0$. More precisely, the recursion for the rescaled expectation values

$$\frac{A_n}{\kappa}\,\tilde{p}_{n+1}+B_n\,\tilde{p}_n-\kappa\,C_n\,\tilde{p}_{n-1}=0\;,\quad n\geq 1\;, \tag{30}$$

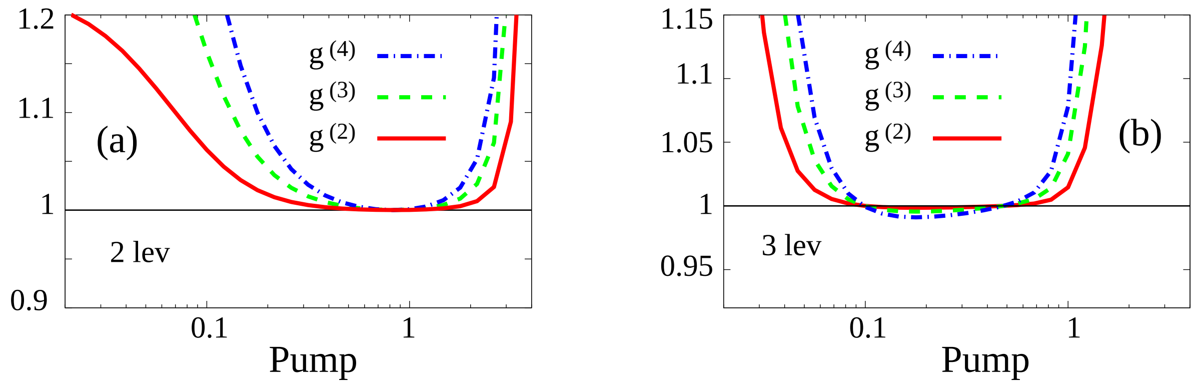

FIG. 5. Second, third and fourth correlation function for (a) the two-level emitter and (b) the three-level emitter. The parameters are the same as in Fig.2 with $\varepsilon = 1$ in (a) and $\varepsilon = 0.004$ in (b).

reduces, in the $\kappa \to 0$ limit, to

$$\tilde{p}_{n+1} = -\kappa \frac{B_0}{A_0} \tilde{p}_n = \nu \, \tilde{p}_n \; , \quad n \geq 1 \; , \tag{31}$$

with the obvious solution $\tilde{p}_n = \nu^{n-1}\tilde{p}_1$. For the values of the pump where $\nu(P)$ is negative only the trivial solution $\tilde{p}_n = 0, \; n \geq 1$ is possible, in order to avoid negative results for positive expectation values. On the other hand, when $\nu(P) > 0$, Eq. (31) holds for $n = 0$ too and one has $\tilde{p}_1 = \nu \, \tilde{p}_0 = \nu$. Then the solution is exponential $\tilde{p}_n = \nu^n, \; n \geq 0$ in accordance with the Poisson statistics.

As κ approaches 0, the product $n\kappa$ in the coefficients of the recursion relation vanishes, and this is how their n-dependence is lost. In the process it is the low-index coefficients that are the first to get close to their limit values, because the limit requires the product $n\kappa$ to be small. Therefore the Poissonian condition $g^{(n)} = 1$ is obeyed by $g^{(2)}$ first, and by $g^{(3)}, g^{(4)}, \ldots$ only later. This is numerically confirmed, as seen in Fig.5, and shows that using $g^{(2)}) = 1$ as a criterion of truly coherent light may be, in this sense, somewhat premature.

IV. CONCLUSION

By solving the master equation for a single emitter in JC interaction with a cavity mode one observes a sudden change in the behavior of the steady-state solution. This is indicative of the onset of lasing and offers the possibility of identifying the conditions for a sharp tran-

sition to a pure, as opposed to approximate, Poissonian statistics. We have shown that these conditions imply an asymptotic regime for the parameters controlling the generation and loss of cavity photons. Specifically, the domain of parameters for which a sharp transitions occurs is defined by both the cavity loss κ and the JC coupling g going to 0, provided that g scales like $\sqrt{\kappa}$.

The result is supported by numerical data, as exemplified for a two-level and a three-level emitter, and is proven using analytical methods for the two-level model. In a previous paper[17], the same scaling limit was shown to give rise to a sharp transition and to reproduce the threshold value known in the literature, for the random injection model of Scully and Lamb. It should be noted that the Scully-Lamb model does not belong to the class considered here: while the latter are "embedded" JC systems the former is rather an "intermittent" JC one. This fact, together with the numerical evidence, suggests that our scaling limit result has a range of validity that is larger than the set of cases for which a full analytic proof is available now.

---

[1] V. DeGiorgio and M. O. Scully, Phys. Rev. A **2**, 1170 (1970)

[2] R. Graham and H. Haken, Z. Physik **237**, 31 (1970)

[3] S. Grossmann and P. H. Richter, Z. Physik **242**, 458 (1971)

[4] M. O. Scully and W. E. Lamb, Phys. Rev. **159**, 208 (1967)

[5] S. Stenholm, Phys. Rep. **6**, 1 (1973)

[6] D. F. Walls and G. J. Milburn, *Quantum Optics* (Springer, Berlin, 1994)

[7] M. Orszag, *Quantum Optics* (Springer, Berlin, Heidelberg, 2000)

[8] G. S. Agarwal and S. D. Gupta, Phys. Rev. A **42**, 1737 (1990)

[9] T. B. Karlovich and S. Y. Kilin, Opt. Spectrosc. **91**, 343 (2001)

[10] A. D. Boozer, Phys. Rev. A **78**, 053814 (2008)

[11] P. R. Rice and H. J. Carmichael, Phys. Rev. A **50**, 4318 (1994)

[12] E. del Valle, F. P. Laussy, and C. Tejedor, Phys. Rev. B **79**, 235326 (2009)

[13] E. del Valle and F. P. Laussy, Phys. Rev. A **84**, 043816 (2011)

[14] A. Auffèves, D. Gerace, J.-M. Gérard, M. F. Santos, L. C. Andreani, and J.-P. Poizat, Phys. Rev. B **81**, 245219 (2010)

[15] Coherent excitation can be described by an additional Hamiltonian term[16], but we will not discuss this approach here.

[16] Y. Mu and C. M. Savage, Phys. Rev. A **46**, 5944 (1992)

[17] P. Gartner, Phys. Rev. A **84**, 053804 (2011)

[18] The steady-state results depend only on the ratio of the parameters, therefore their units are irrelevant and not specified.

[19] E. del Valle and F. P. Laussy, Phys. Rev. Lett. **105**, 233601 (2010)

[20] L. Lorentzen and H. Waadeland, *Continued fractions with applications* (North-Holland, Amsterdam, 1992)

[21] H. J. Carmichael, *Statistical Methods in Quantum Optics 1*, 2nd ed. (Springer, Berlin, Heidelberg, 2002)